CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION DE PRINTEMPS

MATHEMATICS

(Duration : 2 hours)

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This exam is composed of four exercises. One can solve them *in any order*. It is not needed to solve all the exercises to obtain the best possible mark.

Generally speaking, the difficulty of the questions is an increasing function of their number so that *the final questions are more difficult*. It is therefore not recommended to spend too much time on the final questions of an exercise before having solved the first questions of the others.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove)*.

All statements must be *clearly* and *completely* justified.

I. Matrices with a robust diagonal

A matrix $M \in \mathcal{M}_n(\mathbb{C})$ is said to have a robust diagonal if the elements on its diagonal coincide with its eigenvalues (with the same multiplicities). We denote by $\mathcal{D}_n$ the subset of $\mathcal{M}_n(\mathbb{C})$ composed of all the matrices with a robust diagonal.

1) Identify $\mathcal{D}_1$.

2) Show that every matrix in $\mathcal{M}_n(\mathbb{C})$ is similar to a matrix with a robust diagonal.

3) Identify $\mathcal{D}_2$. Is this set open ? closed ? path-connected ? convex ?

4) For a given integer n, is the set of all matrices with a robust diagonal $\mathcal{D}_n$ a vector space ?

5) Determine, among the real symmetric matrices, those belonging to $\mathcal{D}_n$. (Hint : one may introduce the quantity $\mathrm{tr}({}^tMM)$).

II. Study of a sequence

Let c be a non-negative real number and $(u_n)_{n\geq 1}$ be the sequence defined by $u_1 = 1$ and the induction relation

$$u_{n+1} = \sqrt{u_n + cn}.$$

1) Find a real number σ such that for any integer $n \geq 1$, one has $u_n \leq \sigma\sqrt{n}$.

2) Find an equivalent of u_n, when n tends to infinity, of the form $u_n \sim \alpha n^\beta$ (where α and β are real numbers).

3) Compute the limit of $u_n - \alpha n^\beta$ as n tends to infinity.

III. Homogeneous polynomials

For a given positive integer n, we denote by $\mathcal{H}_n \subset \mathbb{R}[X,Y]$ the set of homogeneous polynomials with real coefficients of degree n in two variables (X and Y). We recall that these are the polynomials P having the property that $P(\lambda X, \lambda Y) = \lambda^n P(X,Y)$ for any real number λ.

We denote by $\mathcal{D}_n$ the subset of $\mathcal{H}_n$ composed of the polynomials divisible by $X^2 + Y^2$ and we denote by $\mathcal{L}_n$ the subset of $\mathcal{H}_n$ composed of those polynomials P satisfying

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = 0.$$

1) Prove that, for any integer n, $\mathcal{H}_n$, $\mathcal{D}_n$ and $\mathcal{L}_n$ are vector spaces.

2) Prove that any polynomial in $\mathcal{H}_n$ can be written as

$$\sum_{k=0}^{n} c_k X^k Y^{n-k}$$

where the c_k's are real numbers. What is the dimension of $\mathcal{H}_n$?

3) What are the relations that the coefficients of P must satisfy if $P \in \mathcal{L}_n$?

4) Using question 3), give a basis of $\mathcal{L}_n$.

5) Prove that $\mathcal{D}_n \cap \mathcal{L}_n = \{0\}$.

6) Prove that $\dim \mathcal{D}_n = \dim \mathcal{H}_{n-2}$.

7) Prove that $\mathcal{D}_n \oplus \mathcal{L}_n = \mathcal{H}_n$.

IV. Study of a power series

1) For which integers n does the equality $\sin(n\pi\sqrt{5}) = 0$ hold ?

Let R be the radius of convergence of the power series $\displaystyle\sum_{n\geq 1} \frac{z^n}{\sin(n\pi\sqrt{5})}$.

2) Prove that $R \leq 1$.

3) Prove that, for all $t \in [0, \pi/2]$, one has $\sin t \geq t - t^3/6$.

4) Prove that for any integers p and q, non simultaneously equal to zero, one has

$$|p\sqrt{5} - q| \geq \frac{1}{p\sqrt{5} + q}.$$

5) Using questions 3) and 4), find a real number $c > 0$ such that, for all integers $n \geq 1$, one has

$$|\sin(n\pi\sqrt{5})| \geq \frac{c}{n}.$$

6) Compute R.