ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION AUTOMNE 2015

MATHEMATICS

(Duration : 2 hours)

⋆ ⋆ ⋆

This exam is composed of 5 exercises. One can solve them in *any order*. Generally speaking, the first questions are simpler and the final questions are more difficult. One should therefore *not spend too much time* on these final questions before having solved the other exercices.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove)*. All statements must be *clearly* and *completely* justified.

I.

Let P be a polynomial in $\mathbb{R}[X]$ satisfying $(*)$: $P(P(X)) = P(X)^2$.

1. Are there constant polynomials solutions to $(*)$?

From now on, we assume that P is a non-constant polynomial solution to $(*)$.

2. Determine the degree of P.

3 What is the value of the coefficient of X^2 in P ?

4. Find all the polynomials P solutions to $(*)$?

5. By an analogous reasoning, find the solutions $P \in \mathbb{C}[X]$ of the equation :

$$P(P''(X)) = (P(X+1) - P(X) - P'(X))^3 .$$

II.

Let n be a positive integer, E be an n-dimensional $\mathbb{C}$-vector space and $(e_1, \dots, e_n)$ be a basis of E. Let u be the endomorphism of E defined by the relations :

$$\text{for all } i \in \{1, \dots, n\} : u(e_i) = e_i + f$$

where $f = \sum_{k=1}^{n} e_k$.

1. What is the matrix of u in the basis $(e_1, \ldots, e_n)$? We call it U in the following questions.

2. Is the matrix U invertible ?

3. Let J be the matrix which has ones in all positions ($J_{i,j} = 1$ for all $1 \leq i,j \leq n$). Compute J^2.

4. Find the eigenvalues of U. For each of them, give a basis of the associated eigenspace.

5. Compute, for each non-negative integer m, the m-th power of U as a function of J.

III.

Let M_0 be the 3×3 square matrix :

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

1. Compute ${M_0}^3$ in terms of M_0.

Let now M be a 3×3 square matrix with real coefficients, $M \in \mathcal{M}(\mathbb{R}^3)$, such that

$$M^3 = -M .$$

We assume $M \neq 0$.

2. Is the matrix M diagonalizable as a real matrix? What about diagonalizability as a complex matrix?

From now on, we are only interested in the reduction of M as a real matrix.

3. Show that

$$\mathbb{R}^3 = \text{Ker } M \oplus \text{Ker } (M^2 + I) ,$$

where I stands for the identity matrix in dimension 3.

4. Prove that the dimension of Ker $(M^2 + I)$ is even. Then deduce dim Ker $M = 1$.

5. Show that there exists a vector x in $\mathbb{R}^3$ such that the family $\{x, Mx\}$ is free.

6. Prove that M is similar to M_0.

IV.

In this exercise, we are interested in the solutions to the equation (E) : $\tan x = x$.

1. Show that for each integer n, (E) has exactly one solution in the interval $(n\pi - \pi/2, n\pi + \pi/2)$. We denote it by x_n.

2. Prove that, when n tends to infinity, $x_n - n\pi$ tends towards $\pi/2$.

3. Compute the limit of the quantity $v_n = n(x_n - n\pi - \pi/2)$ when n tends to infinity.

V.

We define the functions

$$I(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \cos tx \, dt \qquad \text{and} \qquad J(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \sin tx \, dt \, .$$

1. Explain why I and J are well defined on $\mathbb{R}$.

2. Show that I and J are differentiable and compute their derivatives I' and J'.

3. Integrating by parts, prove that

$$I'(x) = -\frac{1}{2}J(x) - xJ'(x) \, .$$

In the same way, find a relation between J', I and I'.

4. Deduce that I and J both satisfy a differential equation of the type

$$2(1 + x^2)y' + xy = f$$

where f is a certain function (not necessarily the same for I and J).

5. Solve the preceding system and determine I and J in terms of x uniquely.

* *
*